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SOME APPLICATIONS OF STATIONARY REFLECTION IN $\mathcal{P}_\kappa\lambda$

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ABSTRACT. The priniciple of Stationary Reflection in $\mathcal{P}_\kappa\lambda$ has been successful in the case $\kappa = \omega_1$. Unfortunately the principle fails if $\kappa > \omega_1$. Nonetheless some weaker versions are consistent relative to the existence of some large cardinals. This paper presents some (hopefully) nontrivial applications of these principles and state related problems.

1. INTRODUCTION

This paper is concerned with some combinatorinal statements that hold in the Levy collapse. More specifically we are interested in generalizing or improving the following two results:

Theorem 1 (Foreman–Magidor–Shelah [8]). *The club filter on ω_1 is presaturated after a supercompact cardinal is Levy collapsed to ω_2.*

Theorem 2 (Baumgartner [1]). *Every club subset of $\mathcal{P}_{\omega_2}\omega_3$ has size at least 2^{ω_1} after an ω_1-Erdős cardinal is Levy collapsed to ω_3 (and then Cohen subsets of ω_1 are added).*

Henceforth κ denotes a regular uncountable cardinal. Goldring generalized and refined Theorem 1:

Theorem 3 (Goldring [13]). *The club filter on $\mathcal{P}_{\omega_1}\kappa$ is presaturated after a Woodin cardinal $> \kappa$ is Levy collapsed to κ^+.*

We get a further generalization:

Theorem 4 (Shioya [25]). *The club filter on $\mathcal{P}_\mu\kappa$ is weakly presaturated below $\{x \in \mathcal{P}_\mu\kappa : \operatorname{cf}\sup x = \omega\}$ for every regular uncountable $\mu \leq \kappa$ after a supercompact cardinal $> \kappa$ is Levy collapsed to κ^+.*

These are the subject of §4.

As to Theorem 2 the following generalization is almost immdeate:

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As to Theorem 2 the following generalization is almost immdeate:

Proposition 1. *Every club subset of $\mathcal{P}_\kappa\kappa^+$ has size at least 2^{ω_1} after a supercompact cardinal $> \kappa$ is Levy collapsed to κ^+ and then Cohen subsets of ω_1 are added.*

Building on recent advances in constructing diamonds on $\mathcal{P}_\kappa\lambda$, we get a much stronger result:

Theorem 5 (Shioya [26]). *$\mathcal{P}_\kappa\lambda$ carries a diamond for every $\lambda > \kappa$ after a supercompact cardinal $> \kappa$ is Levy collapsed to κ^+ and then Cohen subsets of ω_1 are added.*

These matters are taken up in §5.

In [8] the principle SR (for Stationary Reflection) was introduced:

SR in $\mathcal{P}_{\omega_1}\lambda$ holds iff for every stationary $S \subset \mathcal{P}_{\omega_1}\lambda$
there is $\omega_1 \subset X$ of size ω_1 s.t. $S \cap \mathcal{P}_{\omega_1}X$ is stationary in $\mathcal{P}_{\omega_1}X$.

Proposition 2 (Foreman–Magidor–Shelah [8]). SR *in $\mathcal{P}_{\omega_1}\lambda$ holds for every $\lambda \geq \omega_2$ after a supercompact cardinal is Levy collapsed to ω_2.*

SR in $\mathcal{P}_{\omega_1}\lambda$ abstracts substantial combinatorics of the model:

Proposition 3 (Todorčević [3]). SR *in $\mathcal{P}_{\omega_1}\lambda$ implies that*

1. *the club filter on ω_1 is presaturated, and*
2. *Chang's Conjecture holds.*

Henceforth it is understood that λ is sufficiently large whenever SR is assumed.

Proposition 3 (2) corresponds to the following

Theorem 6 (Baumgartner [1], Donder–Levinski [5]). *Chang's Conjecture holds after an ω_1-Erdős cardinal is Levy collapsed to ω_2.*

Our proofs of Theorems 4 and 5 are based on suitable generalizations of SR to higher cardinals. One would come up with the following version immediately:

SR in $\mathcal{P}_\kappa\lambda$ holds iff for every stationary $S \subset \mathcal{P}_\kappa\lambda$
there is $\kappa \subset X$ of size κ s.t. $S \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X$.

Unfortunately

Theorem 7 (Shelah–Shioya [23]). SR *in $\mathcal{P}_\kappa\lambda$ fails if $\omega_1 < \kappa < \lambda$.*

Instead we introduce in §3 two principles κ-SR in $\mathcal{P}_{\omega_1}\lambda$ and σ-SR in $\mathcal{P}_\kappa\lambda$ which hold after a supercompact cardinal $> \kappa$ is Levy collapsed to κ^+.

2. Preliminaries

For background material we refer the reader to [17]. Unless otherwise stated κ denotes a regular cardinal $> \omega$ and λ a cardinal $> \kappa$.

Let $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$. For $x \subset \lambda$ let $\mathrm{cl}_f\, x$ be the closure of x under f, i.e. the smallest set $z \subset \lambda$ such that $x \subset z$ and $f``[z]^{<\omega} \subset \mathcal{P}(z)$. We denote the set $\{z \subset \lambda : \mathrm{cl}_f\, z = z\}$ by $C(f)$. It is well-known that the club filter on $\mathcal{P}_\kappa\lambda$ is generated by the sets of the form $\mathcal{P}_\kappa\lambda \cap C(f)$. A set of the form $\mathcal{P}_\kappa\lambda \cap C(f)$ is called σ-club if $f : [\lambda]^{<\omega} \to \mathcal{P}_{\omega_1}\lambda$. It is also known that the club filter on $\mathcal{P}_\kappa\lambda$ is generated by the σ-club sets together with the set $\{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \in \kappa\}$.

Let $\mu < \nu$ be both regular. Denote the set $\{\gamma < \nu : \mathrm{cf}\,\gamma = \mu\}$ by S^μ_ν. Recall from [21] that a club guessing sequence on S^μ_ν is a map $: \gamma \in S^\mu_\nu \mapsto c_\gamma$ an unbounded subset of γ of order type μ such that if $D \subset \nu$ is club, $\{\gamma \in S^\mu_\nu : c_\gamma \subset D\}$ is stationary in ν. Gitik and Shelah [11] constructed such a sequence even with an additional property (see [24] for a simpler proof):

Lemma 1. *Let $\mu < \kappa < \nu$ be all regular. Then there is a club guessing sequence $\langle c_\gamma : \gamma \in S^\mu_\nu\rangle$ such that if $\gamma \in S^\mu_\nu \cap \lim\{\alpha < \nu : \mathrm{cf}\,\alpha \geq \kappa\}$, $c_\gamma \subset \{\alpha < \gamma : \mathrm{cf}\,\alpha \geq \kappa\}$.*

Here $\lim X$ denotes the set of limit points of X.

For later purposes we present a proof of Proposition 4 due to Foreman–Todorčević [9].

Let $\langle c_\gamma : \gamma \in S^\omega_{\omega_2}\rangle$ be a club guessing sequence. Fix $x \in \mathcal{P}_{\omega_1}\omega_2$ such that $\gamma = \sup x$ has cofinality ω. List c_γ in increasing order as $\{\gamma_n : n < \omega\}$. Set $r(x) = \{n < \omega : x \cap (\gamma_{n+1} - \gamma_n) \neq \emptyset\} \in [\omega]^\omega$.

Proposition 4. *$\{x \in \mathcal{P}_{\omega_1}\omega_2 : r(x) = r\}$ is stationary for $r \in [\omega]^\omega$.*

Define a tree order $\leq$ on $[\omega_2]^{<\omega}$ by end-extension. Let T be a subtree of $[\omega_2]^{<\omega}$. For $a \in T$ set $\mathrm{suc}_T(a) = \{\alpha < \omega_2 : a \leq a \cup \{\alpha\} \in T\}$. T is called stationary if $\mathrm{suc}_T(a)$ is stationary in ω_2 for every $a \in T$. $[T]$ denotes the set of infinite branches through T. For $a \in T$ let $T^a = \{b - a : a \leq b \in T\}$.

The following lemma is from [24]:

Lemma 2. *Let T be a stationary subtree of $[S^\kappa_\nu]^{<\omega}$ and $F : T \to \mathcal{P}_\kappa\lambda$. Then there are a stationary subtree T^* of T and $h : T^* \to \nu$ such that if $a < b \in T^*$, $F(b) \cap \min(b - a) \subset h(a)$.*

Proof of Proposition 4. Let $f : [\omega_2]^{<\omega} \to \mathcal{P}_{\omega_1}\omega_2$. It suffices to give a countable $x \in C(f)$ such that $\mathrm{cf}\sup x = \omega$ and $r(x) = r$.

By Lemma 2 we have a stationary subtree T^* of $[S^{\omega_1}_{\omega_2}]^{<\omega}$ and a map $h : T^* \to \omega_2$ such that if $a < b \in T^*$, $(\mathrm{cl}_f\, b) \cap \min(b - a) \subset h(a)$. Then

$D = \{\gamma < \omega_2 : \mathrm{cl}_f\, \gamma = \gamma \wedge \forall a \in T^* \cap [\gamma]^{<\omega}(h(a) < \gamma \in \lim \mathrm{suc}_{T^*}(a))\}$ is club. Take $\gamma \in S^\omega_{\omega_2} \cap D$ with $c_\gamma \subset D - \omega_1$. List c_γ in increasing order as $\{\gamma_n : n < \omega\}$, and r as $\{n(k) : k < \omega\}$.

By induction on $k < \omega$ we choose $\gamma_{n(k)} < \alpha_k < \gamma_{n(k)+1}$ so that $\{\alpha_i : i < k\} \in T^*$ as follows:

Assume we have $\{\alpha_i : i < k\}$. Since $\{\alpha_i : i < k\} \in T^* \cap [\gamma_{n(k)+1}]^{<\omega}$ and $\gamma_{n(k)+1} \in D$, we have $\gamma_{n(k)} < \alpha_k \in \gamma_{n(k)+1} \cap \mathrm{suc}_{T^*}\{\alpha_i : i < k\}$. Then $\{\alpha_i : i \leq k\} \in T^*$, as desired.

Set $x = \bigcup_{k<\omega} \mathrm{cl}_f\{\alpha_i : i \leq k\} \in \mathcal{P}_{\omega_1}\omega_2 \cap C(f)$. We have $x \subset \mathrm{cl}_f\, \gamma = \gamma$ by $\{\alpha_i : i < \omega\} \subset \gamma \in D$. Hence $\sup x = \gamma$ by $\sup_{i<\omega} \alpha_i = \sup_{i<\omega} \gamma_{n(i)} = \gamma$.

Claim. $r(x) = \{n(k) : k < \omega\}$.

Proof. Since $\gamma_{n(k)} < \alpha_k < \gamma_{n(k)+1}$, we have $n(k) \in r(x)$ for every $k < \omega$. To see the converse, fix $k < l < \omega$. Then by the choice of h we have $(\mathrm{cl}_f\{\alpha_i : i \leq l\}) \cap \gamma_{n(k+1)} \subset (\mathrm{cl}_f\{\alpha_i : i \leq l\}) \cap \alpha_{k+1} \subset h(\{\alpha_i : i \leq k\})$. Since $\{\alpha_i : i \leq k\} \in T^* \cap [\gamma_{n(k)+1}]^{<\omega}$ and $\gamma_{n(k)+1} \in D$, we have $h(\{\alpha_i : i \leq k\}) < \gamma_{n(k)+1}$. Hence $(\mathrm{cl}_f\{\alpha_i : i \leq l\}) \cap \gamma_{n(k+1)} \subset \gamma_{n(k)+1}$. Thus $x \cap \gamma_{n(k+1)} \subset \gamma_{n(k)+1}$, as desired. □

This completes the proof. □

When applying SR, some "catching-tails" lemma is needed. The prototype can be found in Shelah's proof [19] of Chang's conjecture in the Levy collapse of a measurable cardinal to ω_2. It was extensively exploited in [8]. The following version is used in §5 together with κ-SR in $\mathcal{P}_{\omega_1}\lambda$:

Lemma 3. *Let D be σ-club in $\mathcal{P}_\kappa 2^\theta$ with $\theta = 2^{\kappa^+}$. Then there is $d : [2^\theta]^{<\omega} \to \mathcal{P}_{\omega_1} 2^\theta$ such that if $z \in \mathcal{P}_\kappa 2^\theta \cap C(d)$ and $u \in \mathcal{P}_\kappa \kappa^+$, $z \in D$ and $\mathrm{cl}_d(z \cup u) \cap \theta = \mathrm{cl}_d((z \cap \theta) \cup u) \cap \theta$.*

The following variation is used in §4 together with σ-SR in $\mathcal{P}_\kappa\lambda$:

Lemma 4. *Let D be σ-club in $\mathcal{P}_\kappa 2^\lambda$ with $\lambda = 2^\nu$ and $\nu > \kappa$. List the functions $: \nu \to \mathcal{P}_{\omega_1}\lambda$ as $\{h_\alpha : \alpha < \lambda\}$. Then there is $d : {}^{<\omega} \to \mathcal{P}_{\omega_1} 2^\lambda$ such that if $z \in \mathcal{P}_\kappa 2^\lambda \cap C(d)$ and $\xi < \nu$, $z \in C(d)$ and $\mathrm{cl}_d(z \cup \{\xi\}) \cap \lambda = \bigcup\{e_\alpha(\xi) : \alpha \in z \cap \lambda\} = \mathrm{cl}_d((z \cap \lambda) \cup \{\xi\}) \cap \lambda$.*

3. Principles of Stationary Reflection

This section presents two generalizations of SR in $\mathcal{P}_{\omega_1}\lambda$ to higher cardinals. Here is the weaker version:

κ-SR in $\mathcal{P}_{\omega_1}\lambda$ holds iff for every stationary $S \subset \mathcal{P}_{\omega_1}\lambda$

there is $\kappa \subset X$ of size κ s.t. $S \cap \mathcal{P}_{\omega_1} X$ is stationary in $\mathcal{P}_{\omega_1} X$.

So SR in $\mathcal{P}_{\omega_1}\lambda$ from [8] is just ω_1-SR in $\mathcal{P}_{\omega_1}\lambda$ in our sense.

Proposition 5. *κ-SR in $\mathcal{P}_{\omega_1}\lambda$ holds for every $\lambda \geq \kappa^+$ after a supercompact cardinal $> \kappa$ is Levy collapsed to κ^+ (and then Cohen subsets of ω_1 are added).*

Crucial to the proof is the following well-known fact:

Lemma 5 (Shelah [19]). *Every stationary subset of $\mathcal{P}_{\omega_1}\lambda$ remains stationary after forcing with a countably closed poset.*

Now assume $\lambda^{<\kappa} = \lambda$. Fix a bijection $\varphi : {}^{<\kappa}\mathcal{P}_\kappa\lambda \to \lambda$. Define

$$S^{\varphi}_{\kappa\lambda} = \{x \in \mathcal{P}_\kappa\lambda : \exists\delta < \kappa\exists t : \delta \to \mathcal{P}_\kappa\lambda$$
$$(x \subset \bigcup t``\delta \wedge \sup\{\eta < \delta : \varphi(t|\eta) \in x\} = \delta)\}.$$

It is easily seen that $S^{\varphi}_{\kappa\lambda}$ is σ-stationary in $\mathcal{P}_\kappa\lambda$. We often write $S_{\kappa\lambda}$ for $S^{\varphi}_{\kappa\lambda}$. This causes no confusion because $S^{\varphi}_{\kappa\lambda}$ and $S^{\psi}_{\kappa\lambda}$ agree on a σ-club set if $\psi : {}^{<\kappa}\mathcal{P}_\kappa\lambda \to \lambda$ is another bijection.

Lemma 5 is generalized as follows:

Lemma 6. *If $\lambda^{<\kappa} = \lambda$, every σ-stationary subset of $S_{\kappa\lambda}$ remains σ-stationary in $\mathcal{P}_\kappa\lambda$ after forcing with a κ-closed poset.*

$S_{\kappa\lambda}$ is maximal with respect to this preoperty:

Lemma 7. *If $\lambda^{<\kappa} = \lambda$, $S_{\kappa\lambda}$ has a σ-club subset of $\mathcal{P}_\kappa\lambda$ after forcing with a κ-closed poset which collapses $|\lambda|$ to κ.*

It is essential to assume $\lambda^{<\kappa} = \lambda$ in our arguments. So we ask:

Question 1. *Let $\mathrm{cf}\,\lambda < \kappa$. Does there exist a stationary $S \subset \mathcal{P}_\kappa\lambda$ such that every stationary subset of S remains stationary after forcing with a κ-closed poset?*

We have the expected definition and results:

σ-SR in $\mathcal{P}_\kappa\lambda$ holds iff for every σ-stationary $S \subset S_{\kappa\lambda}$
there is $\kappa \subset X$ of size κ s.t. $S \cap \mathcal{P}_\kappa X$ is σ-stationary in $\mathcal{P}_\kappa X$.

Proposition 6. *σ-SR in $\mathcal{P}_\kappa\lambda$ holds for every $\lambda > \kappa$ with $\lambda^{<\kappa} = \lambda$ after a supercompact cardinal $> \kappa$ is Levy collapsed to κ^+.*

Proposition 7. *If $\lambda^{<\kappa} = \lambda$, σ-SR in $\mathcal{P}_\kappa\lambda$ implies κ-SR in $\mathcal{P}_{\omega_1}\lambda$.*

To ask one more question, let us restate Theorem 7 in more detail:

Theorem 8 (Shelah–Shioya [23]). *Let $\omega_1 < \kappa < \lambda$. Then there is a stationary subset of $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(x \cap \kappa) = \omega_1 \wedge \mathrm{cf}\sup(x \cap \kappa^+) = \omega\}$ that does not reflect.*

Question 2. *Let $\omega_1 < \kappa < \lambda$. Does there exist a stationary subset of $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(x \cap \kappa) = \omega \wedge \mathrm{cf}\sup(x \cap \kappa^+) = \omega_1\}$ that does not reflect?*

4. Applications I: Weak forms of saturation

The standard argument shows that in the model of Theorem 1 $\diamondsuit_{\omega_1}$ holds, so the club filter on ω_1 is not ω_2-saturated. This fact motivates the following definition due to Baumgartner–Taylor [2]:

A normal filter F on $\mathcal{P}_\kappa\lambda$ is presaturated iff $\Vdash_{F^+}$ "every set of ordinals of size $< \kappa$ can be covered by a set of size λ in V".

In fact the original definition of presaturation was somewhat weaker. The current definition has a natural reformulation in combinatorial terms like Lemma 8 below. In any case a λ^+-saturated normal filter on $\mathcal{P}_\kappa\lambda$ is presaturated.

The proof of Theorem 3 would be substantially simplified if one uses Propositions 5 and 8. The price is that we must assume the existence of a supercompact cardinal.

Proposition 8. *κ-SR in $\mathcal{P}_{\omega_1}\lambda$ implies that the club filter on $\mathcal{P}_{\omega_1}\kappa$ is presaturated.*

It is known to be impossible to generalize Theorem 3 in a straightfoward way:

Theorem 9 (Shelah [20], Burke–Matsubara [4]). *The club filter on $\mathcal{P}_\mu\kappa$ is not presaturated if $\mu = \kappa$ is a successor cardinal $> \omega_1$ or if $\mu < \kappa$ are both regular uncountable.*

With the following result Gitik "almost" closed the matter:

Theorem 10 (Gitik [10]). *It is consistent that the club filter on an inaccessible cardinal is presaturated.*

Gitik's argument, however, involves square sequences. So one can still ask:

Question 3. *Is it consistent that the club filter on a supercompact cardinal is presaturated?*

By Theorem 9 the club filter on ω_2 cannot be presaturated. Gitik and Shelah asked whether it can have a weaker property. The following definition is due to them:

A normal filter F on $\mathcal{P}_\kappa\lambda$ is weakly presaturated iff $\Vdash_{F^+}$ "every countable set of ordinals can be covered by a set of size λ in V".

Theorem 11 (Gitik–Shelah [12]). *It is consistent that the club filter on ω_2 is weakly presaturated.*

The following question seems to be open:

Question 4. *Is it consistent that the club filter on $\mathcal{P}_{\omega_2}\omega_3$ is weakly presaturated?*

Theorem 4 follows from Proposition 6 and the following

Theorem 12 (Shioya [25]). *σ-SR in $\mathcal{P}_\kappa\lambda$ implies that the club filter on $\mathcal{P}_\mu\kappa$ is weakly presaturated below $\{x \in \mathcal{P}_\mu\kappa : \operatorname{cf}\sup x = \omega\}$ for every regular uncountable $\mu \leq \kappa$.*

It is almost certain that much less than a supercompact suffices for a direct proof of Theorem 4, which should be much more complicated than that of Theorem 12. The proof of Theorem 12 combines those of Proposition 8 and the following

Proposition 9. *σ-SR in $\mathcal{P}_\kappa\lambda$ implies that the club filter on $\mathcal{P}_\mu\kappa$ is precipitous for every regular uncountable $\mu \leq \kappa$.*

Recall from [16] that

F is precipitous iff $\Vdash_{F^+}$ "the generic ultrapower is well-founded".

It is easy to see that a weakly presaturated filter is precipitous and moreover the generic ultrapower is closed under countable sequences.

Proposition 9 corresponds to the following

Theorem 13 (Goldring [14]). *The club filter on $\mathcal{P}_\mu\kappa$ is precipitous for every regular uncountable $\mu \leq \kappa$ after a supercompact cardinal $> \kappa$ is Levy collapsed to κ^+.*

Prior to this, Foreman–Magidor–Shelah [8] had proved that the club filter on $\mathcal{P}_\mu\kappa$ is precipitous below some stationary set in the same model. The set is the projection of IA, the set of Internally Approachable sets. Goldring observed that IA can be replaced by its technical enlargement IA*. In fact IA* had already been introduced in [8] in a similar context. Our $S_{\kappa\lambda}$ from §3 is a combinatorial reformulation of IA*.

As a sample of an application of σ-SR we present a proof of Proposition 9. Before proceeding let us recall

Lemma 8 (Jech–Prikry [16]). *F is precipitous iff for every $S \in F^+$ and a sequence $\{A_n : n < \omega\}$ of maximal antichains in F^+ below S such that A_{n+1} refines A_n there is a descending sequence $\{S_n : n < \omega\}$ such that $S_n \in A_n$ and $\bigcap_{n<\omega} S_n \neq \emptyset$.*

Proof of Proposition 9. As noted in [14], it suffices to show that the filter $C^\sigma_{\kappa\kappa}$ on $\mathcal{P}_\kappa\kappa$ generated by the σ-club sets is precipitous.

Set $\nu = 2^{2^{<\kappa}}$ and $\lambda = 2^\nu$. Fix a σ-stationary $S \subset \mathcal{P}_\kappa\kappa$. We can assume S is co-σ-stationary as well. For $n < \omega$ let $\{S_{n,\xi} : \xi < \nu\}$ be a maximal antichain in $(C^\sigma_{\kappa\kappa})^+$ below S such that for $\xi < \nu$ there is $\zeta < \nu$

with $S_{n+1,\xi} \le S_{n,\zeta}$. Set $\mathrm{suc}_n(\zeta) = \{\xi < \nu : S_{n,\xi} \le S_{n-1,\zeta}\}$ for $n < \omega$ and $\zeta < \nu$. We stipulate $S_{-1,\zeta} = S$. Then $\{S_{n,\xi} : \xi \in \mathrm{suc}_n(\zeta)\}$ is a maximal antichain in $(\mathcal{C}^\sigma_{\kappa\kappa})^+$ below $S_{n-1,\zeta}$. We use σ-SR to give $x \in \mathcal{P}_\kappa\kappa$ and $\{\xi_n : n < \omega\} \subset \nu$ such that $x \in S \cap S_{n,\xi_n}$ and $\xi_n \in \mathrm{suc}_n(\xi_{n-1})$ for every $n < \omega$.

Let $\{g_\gamma : \gamma < 2^\lambda\}$ and $\{h_\alpha : \alpha < \lambda\}$ list the functions of the form $g : [\lambda]^{<\omega} \to \mathcal{P}_{\omega_1}\lambda$ and $h : \nu \to \mathcal{P}_{\omega_1}\lambda$ respectively. Fix a bijection $\varphi : {}^{<\kappa}\mathcal{P}_\kappa\lambda \to \lambda$. Set

$$\begin{aligned} D = \{z \in \mathcal{P}_\kappa 2^\lambda : {} & \forall\alpha \in z \cap \lambda(h_\alpha``(z \cap \nu) \subset \mathcal{P}(z)) \wedge \\ & \forall\gamma \in z(z \cap \lambda \in C(g_\gamma)) \wedge \forall\beta \in z \cap \lambda \forall\xi \in z \cap \nu \\ & (\varphi(\langle \bigcup_{\alpha \in \varphi^{-1}(\beta)(\eta)} h_\alpha(\xi) : \eta \in \operatorname{dom}\varphi^{-1}(\beta)\rangle) \in z)\}, \end{aligned}$$

which is σ-club in $\mathcal{P}_\kappa 2^\lambda$. We have $d : [2^\lambda]^{<\omega} \to \mathcal{P}_{\omega_1}2^\lambda$ as in Lemma 4. For $n < \omega$ and $\zeta < \nu$ set

$$\begin{aligned} T_{n,\zeta} = \{x \in \mathcal{P}_\kappa\lambda : {} & \exists\xi \in \mathrm{suc}_n(\zeta) \\ & (\mathrm{cl}_d(x \cup \{\xi\}) \cap \kappa = x \cap \kappa \in S_{n,\xi} \cup (\mathcal{P}_\kappa\kappa - S_{n-1,\zeta}))\}. \end{aligned}$$

Claim. *$S_{\kappa\lambda} \cap C \subset T_{n,\zeta}$ for some σ-club $C \subset \mathcal{P}_\kappa\lambda$.*

Proof. Let $T \subset S_{\kappa\lambda}$ be σ-stationary in $\mathcal{P}_\kappa\lambda$. We show that $T \cap T_{n,\zeta} \ne \emptyset$.

By σ-SR we have $\kappa \subset X \subset \lambda$ of size κ such that $T \cap \mathcal{P}_\kappa X$ is σ-stationary in $\mathcal{P}_\kappa X$. Fix a bijection $\pi : X \to \kappa$. Since

$$\{x \in T \cap \mathcal{P}_\kappa X : \pi``x = x \cap \kappa\}$$

is σ-stationary in $\mathcal{P}_\kappa X$, so is $S' = \{x \cap \kappa : x \in T \cap \mathcal{P}_\kappa X \wedge \pi``x = x \cap \kappa\}$ in $\mathcal{P}_\kappa\kappa$. Since $\{S_{n,\xi} : \xi \in \mathrm{suc}_n(\zeta)\} \cup \{\mathcal{P}_\kappa\kappa - S_{n-1,\zeta}\}$ is a maximal antichain in $(\mathcal{C}^\sigma_{\kappa\kappa})^+$, we have $\xi \in \mathrm{suc}_n(\zeta)$ such that $S' \cap (S_{n,\xi} \cup (\mathcal{P}_\kappa\kappa - S_{n-1,\zeta}))$ is σ-stationary in $\mathcal{P}_\kappa\kappa$. Since $\{y \in \mathcal{P}_\kappa 2^\lambda : y \cap \kappa \in S' \cap (S_{n,\xi} \cup (\mathcal{P}_\kappa\kappa - S_{n-1,\zeta}))\}$ is σ-stationary in $\mathcal{P}_\kappa 2^\lambda$, we have $y \in \mathcal{P}_\kappa 2^\lambda \cap C(d)$ such that $\xi \in y$ and $\pi``(y \cap X) = y \cap \kappa \in S' \cap (S_{n,\xi} \cup (\mathcal{P}_\kappa\kappa - S_{n-1,\zeta}))$. Since $y \cap \kappa \in S'$, we have $x \in T \cap \mathcal{P}_\kappa X$ with $\pi``x = x \cap \kappa = y \cap \kappa$. Then $x = \pi^{-1}``(x \cap \kappa) = \pi^{-1}``(y \cap \kappa) = y \cap X$. Since $x \cup \{\xi\} \subset y \in C(d)$, $\mathrm{cl}_d(x \cup \{\xi\}) \subset y$. Hence $x \cap \kappa \subset \mathrm{cl}_d(x \cup \{\xi\}) \cap \kappa \subset y \cap \kappa = x \cap \kappa \in S_{n,\xi} \cup (\mathcal{P}_\kappa\kappa - S_{n-1,\zeta})$. Thus $x \in T \cap T_{n,\zeta}$, as desired. □

Take $\gamma < 2^\lambda$ with $S_{\kappa\lambda} \cap C(g_\gamma) \subset \bigcap_{n<\omega} \Delta_{\zeta<\nu} T_{n,\zeta}$. Set

$$E = \{z \cap \lambda : z \in \mathcal{P}_\kappa 2^\lambda \cap C(d)\},$$

which is σ-club in $\mathcal{P}_\kappa\lambda$. Let

$$S^*_{\kappa\lambda} = \{x \in \mathcal{P}_\kappa\lambda : \exists\delta < \kappa\exists t : \delta \to E \text{ increasing}$$
$$(x \subset \bigcup t``\delta \wedge \sup\{\eta < \delta : \varphi(t|\eta) \in x\} = \delta)\},$$

a subset of $S_{\kappa\lambda}$. It is easily seen that $S^*_{\kappa\lambda}$ is σ-stationary in $\mathcal{P}_\kappa\lambda$. Set $S^* = \{z \in \mathcal{P}_\kappa 2^\lambda \cap C(d) : \gamma \in z \wedge z \cap \lambda \in S^*_{\kappa\lambda}\}$, which is σ-stationary in $\mathcal{P}_\kappa 2^\lambda$.

Claim. *$\{z \cap \kappa : z \in S^*\}$ is σ-club in $\mathcal{P}_\kappa\kappa$.*

Proof. Build an increasing sequence $t : \kappa \to E$ by

$$t(\delta) = \mathrm{cl}_d(\delta \cup \{\varphi(t|\eta) : \eta < \delta\} \cup \bigcup t``\delta \cup \{\gamma\}) \cap \lambda.$$

Set $Y = \bigcup t``\kappa \subset \lambda$. Then we have $\kappa \cup \{\varphi(t|\eta) : \eta < \kappa\} \subset Y$ and $\mathrm{cl}_d(a \cup \{\gamma\}) \cap \lambda \subset Y$ for every $a \in [Y]^{<\omega}$. Define $f : [Y]^{<\omega} \to \mathcal{P}_{\omega_1}Y$ by $f(a) = \{\varphi(t|\eta) : \eta \in a \cap \kappa\} \cup (\mathrm{cl}_d(a \cup \{\gamma\}) \cap \lambda)$. Note that for every $x \in \mathcal{P}_\kappa Y$ there is $\eta < \kappa$ with $x \subset t(\eta)$. Hence

$$C = \{x \in \mathcal{P}_\kappa\kappa : \forall a \in [x]^{<\omega}\exists\eta \in x(\mathrm{cl}_f\, a \cap \kappa \subset x \wedge \mathrm{cl}_f\, a \subset t(\eta))\}$$

is σ-club in $\mathcal{P}_\kappa\kappa$. We claim that $C \subset \{z \cap \kappa : z \in S^*\}$.

Fix $x \in C$. Set $y = \mathrm{cl}_f\, x \in \mathcal{P}_\kappa Y$. Then $x \subset y \cap \kappa = \mathrm{cl}_f\, x \cap \kappa = \bigcup_{a \in [x]^{<\omega}} \mathrm{cl}_f\, a \cap \kappa \subset x$ by $x \in C$. Set $z = \mathrm{cl}_d(y \cup \{\gamma\}) \in \mathcal{P}_\kappa 2^\lambda \cap C(d)$. Then $y \subset z \cap \lambda = \mathrm{cl}_d(y \cup \{\gamma\}) \cap \lambda = \bigcup_{a \in [y]^{<\omega}} \mathrm{cl}_d(a \cup \{\gamma\}) \cap \lambda \subset \bigcup_{a \in [y]^{<\omega}} f(a) \subset y$ by $y \in C(f)$. Since $x \in C$, $\varphi(t|\eta) \in f(\{\eta\}) \subset \mathrm{cl}_f\, x = y$ for every $\eta \in x$. Hence $\sup\{\eta < \sup x : \varphi(t|\eta) \in y\} = \sup x$. Since $x \in C$, $y = \mathrm{cl}_f\, x = \bigcup_{a \in [x]^{<\omega}} \mathrm{cl}_f\, a \subset \bigcup_{\eta \in x} t(\eta) \subset \bigcup t``\sup x$. Hence $t|\sup x$ witnesses $y \in S^*_{\kappa\lambda}$. Thus $x = z \cap \kappa$ and $z \in S^*$, as desired. □

Build an increasing sequence $\{z_n : n < \omega\} \subset S^*$ and $\{\xi_n : n < \omega\} \subset \nu$ such that $z_{n+1} \cap \kappa = z_0 \cap \kappa \in S \cap S_{n,\xi_n}$ and $\xi_n \in z_{n+1} \cap \mathrm{suc}_n(\xi_{n-1})$ as follows:

Take $z_0 \in S^*$ with $z_0 \cap \kappa \in S$. Assume we have z_n as above. Since $\gamma \in z_n \in \mathcal{P}_\kappa 2^\lambda \cap C(d) \subset D$, $z_n \cap \lambda \in C(g_\gamma) \cap S_{\kappa\lambda} \subset \Delta_{\zeta<\nu} T_{n,\zeta}$. Since $\xi_{n-1} \in z_n \cap \nu$, $z_n \cap \lambda \in T_{n,\xi_{n-1}}$. Hence we have $\xi_n \in \mathrm{suc}_n(\xi_{n-1})$ such that $\mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi_n\}) \cap \kappa = z_n \cap \kappa \in S_{n,\xi_n} \cup (\mathcal{P}_\kappa\kappa - S_{n-1,\xi_{n-1}})$. Since $z_n \cap \kappa \in S_{n-1,\xi_{n-1}}$, $z_n \cap \kappa \in S_{n,\xi_n}$. Set $z_{n+1} = \mathrm{cl}_d(z_n \cup \{\xi_n\})$. Since $z_{n+1} \in C(d)$, $z_{n+1} \cap \lambda = \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi_n\}) \cap \lambda$ by Lemma 1. Hence $z_{n+1} \cap \kappa = \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi_n\}) \cap \kappa = z_n \cap \kappa = z_0 \cap \kappa \in S \cap S_{n,\xi_n}$.

Claim. $z_{n+1} \cap \lambda \in S^*_{\kappa\lambda}$.

Proof. Let $s : \delta^* \to E$ witness $z_n \cap \lambda \in S^*_{\kappa\lambda}$. Define $t : \delta^* \to E$ by $t(\eta) = \mathrm{cl}_d(s(\eta) \cup \{\xi_n\}) \cap \lambda$. We claim that t witnesses $z_{n+1} \cap \lambda \in S^*_{\kappa\lambda}$.

Since s is increasing, so is t. Since $z_n \cap \lambda \subset \bigcup s``\delta^*$, we have

$$\begin{aligned} z_{n+1} \cap \lambda &= \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi_n\}) \cap \lambda \\ &\subset \mathrm{cl}_d(\bigcup s``\delta^* \cup \{\xi_n\}) \cap \lambda \\ &= \bigcup_{\eta<\delta^*} \mathrm{cl}_d(s(\eta) \cup \{\xi_n\}) \cap \lambda \\ &= \bigcup t``\delta^*. \end{aligned}$$

By the choice of d we have $t(\eta) = \bigcup_{\alpha \in s(\eta)} h_\alpha(\xi_n)$ for every $\eta < \delta^*$.

Fix $\delta < \delta^*$ with $\varphi(s|\delta) \in z_n \cap \lambda$. Since $\mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi_n\}) \in C(d) \subset D$, $\varphi(t|\delta) = \varphi(\langle \bigcup_{\alpha \in s(\eta)} h_\alpha(\xi_n) : \eta < \delta \rangle) \in \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi_n\}) \cap \lambda = z_{n+1} \cap \lambda$. Hence

$$\begin{aligned} &\sup\{\eta < \delta^* : \varphi(t|\eta) \in z_{n+1} \cap \lambda\} \\ = &\sup\{\eta < \delta^* : \varphi(s|\eta) \in z_n \cap \lambda\} \\ = &\,\delta^*, \end{aligned}$$

as desired. □

This completes the proof. □

5. Applications II: Diamonds

The following definition is due to Jech [15]:

A map $g : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ is a diamond iff

for every $A \subset \lambda$ the set $\{x \in \mathcal{P}_\kappa\lambda : A \cap x = g(x)\}$ is stationary.

We are interested in diamonds because of the following

Proposition 10 (Jech [15]). *Assume $\mathcal{P}_\kappa\lambda$ carries a diamond. Then*

1. *the club filter on $\mathcal{P}_\kappa\lambda$ is not 2^λ-saturated,*
2. *$\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets, and*
3. *every club subset of $\mathcal{P}_\kappa\lambda$ has size $\lambda^{<\kappa}$.*

Perhaps the first significant result on diamonds on $\mathcal{P}_\kappa\lambda$ is the following

Theorem 14 (Donder–Matet [6]). *$\mathcal{P}_\kappa\lambda$ carries a diamond for every $\lambda > 2^{<\kappa}$.*

See [24] for a correction. More recently Shelah established

Theorem 15 (Shelah [22]). *$\mathcal{P}_{\omega_1}\lambda$ carries a diamond for every $\lambda > \omega_1$.*

Unfortunately we could fill in the details of Shelah's proof only in the case $\lambda = \omega_2$. Let us present it first:

Shelah's proof for $\lambda = \omega_2$. Fix a club guessing sequence $\langle c_\gamma : \gamma \in S^\omega_{\omega_2}\rangle$ and a disjoint family $\{I_n : n < \omega\} \subset [\omega]^\omega$ with $n < \min I_n$. We have a map $\psi(z) : \mathcal{P}(\omega) \to \mathcal{P}(z)$ for $z \in \mathcal{P}_{\omega_1}\omega_2$ such that $\psi(z)``[I_n]^\omega = \mathcal{P}(z)$ for every $n < \omega$.

Fix $x \in \mathcal{P}_{\omega_1}\omega_2$ such that $\gamma = \sup x$ has cofinality ω. List c_γ in increasing order as $\{\gamma_n : n < \omega\}$. Set

$$r(x) = \{n < \omega : x \cap (\gamma_{n+1} - \gamma_n) \neq \emptyset\} \in [\omega]^\omega.$$

Let $g(x) = \bigcup_{n \in r(x)} \psi(x \cap \gamma_n)(I_n \cap r(x))$. We claim that $g : \mathcal{P}_{\omega_1}\omega_2 \to \mathcal{P}_{\omega_1}\omega_2$ works.

Let $A \subset \omega_2$ and $f : [\omega_2]^{<\omega} \to \mathcal{P}_{\omega_1}\omega_2$. It suffices to give a countable $x \in C(f)$ such that $A \cap x = g(x)$.

For $\omega_1 \leq \gamma < \omega_2$ fix a bijection $\pi_\gamma : \omega_1 \to \gamma$. We can assume if $x \in C(f)$, $\pi_\gamma``(x \cap \omega_1) = x \cap \gamma$ for every $\gamma \in x - \omega_1$. For $\delta < \omega_1$ let $T_\delta = \{a \in [S^{\omega_1}_{\omega_2}]^{<\omega} : \mathrm{cl}_f(a \cup \delta) \cap \omega_1 = \delta\}$, which is a (possibly empty) subtree of $[S^{\omega_1}_{\omega_2}]^{<\omega}$.

Claim. *There are a stationary subtree T^* of some T_δ and $h : T^* \to \omega_2$ such that if $a < b \in T^*$, $\mathrm{cl}_f(b \cup \delta) \cap \min(b - a) \subset h(a)$.*

Proof. First we give $\delta < \omega_1$ such that $[T_\delta] \cap [C]^\omega \neq \emptyset$ for every club $C \subset \omega_2$: Suppose to the contrary we have a club $C_\delta \subset \omega_2$ for $\delta < \omega_1$ such that $[T_\delta] \cap [C_\delta]^\omega = \emptyset$. Then $C = \bigcap_{\delta<\omega_1} C_\delta$ is club in ω_2. Take $B \subset S^{\omega_1}_{\omega_2} \cap C$ of order type ω. We have $\delta < \omega_1$ with $\mathrm{cl}_f(B \cup \delta) \cap \omega_1 = \delta$. Then $B \in [T_\delta] \cap [C_\delta]^\omega$. Contradiction.

Hence $T' = \{b \in T_\delta : \forall a \leq b \forall C \subset \omega_2 \text{ club } ([T^a_\delta] \cap [C]^\omega \neq \emptyset)\}$ is a stationary subtree of T_δ as in the proof of Lemma 2. Finally Lemma 2 gives us a stationary subtree T^* of T' and $h : T^* \to \omega_2$ as required above. $\square$

Now

$$D = \{\gamma < \omega_2 : \mathrm{cl}_f\, \gamma = \gamma \wedge \forall a \in T^* \cap [\gamma]^{<\omega}(h(a) < \gamma \in \lim \mathrm{suc}_{T^*}(a))\}$$

is club. Take $\gamma \in S^\omega_{\omega_2} \cap D$ with $c_\gamma \subset D - \omega_1$. List c_γ in increasing order as $\{\gamma_n : n < \omega\}$.

By induction on $k < \omega$ we choose $n(k) < \omega$, $\gamma_{n(k)} < \alpha_k < \gamma_{n(k)+1}$ and $J_k \in [I_{n(k)}]^\omega$ so that $n(k) < n(k+1)$, $\{\alpha_i : i < k\} \in T^*$ and $\psi(\mathrm{cl}_f(\{\alpha_i : i \leq k\} \cup \delta) \cap \gamma_{n(k)})(J_k) = A \cap \mathrm{cl}_f(\{\alpha_i : i \leq k\} \cup \delta) \cap \gamma_{n(k)}$ as follows:

First set $n(0) = 0$. Assume next we have $\{\alpha_i : i < k\}$ and $n(k)$ as above. Since $\{\alpha_i : i < k\} \in T^* \cap [\gamma_{n(k)+1}]^{<\omega}$ and $\gamma_{n(k)+1} \in D$, we have $\gamma_{n(k)} < \alpha_k \in \gamma_{n(k)+1} \cap \mathrm{suc}_{T^*}\{\alpha_i : i < k\}$. Then $\{\alpha_i : i \leq k\} \in T^*$. Since $\psi(\mathrm{cl}_f(\{\alpha_i : i \leq k\} \cup \delta) \cap \gamma_{n(k)})``[I_{n(k)}]^\omega = \mathcal{P}(\mathrm{cl}_f(\{\alpha_i : i \leq k\} \cup \delta) \cap \gamma_{n(k)})$, we have $J_k \in [I_{n(k)}]^\omega$ as required above. Set

$$n(k+1) = \min(\bigcup_{i\le k} J_i - (n(k)+1)) > n(k).$$

Note that $n(k) \le n(l) < \min I_{n(l)} \le \min J_l$ if $k \le l < \omega$. Hence $n(k+1) = \min(\bigcup_{i<\omega} J_i - (n(k)+1))$. Thus

$$\{n(k) : k < \omega\} = \{0\} \cup \bigcup_{k<\omega} J_k.$$

Now set $x = \bigcup_{k<\omega} \mathrm{cl}_f(\{\alpha_i : i \le k\} \cup \delta) \in \mathcal{P}_{\omega_1}\omega_2 \cap C(f)$. Since $\{\alpha_i : i < \omega\} \cup \delta \subset \gamma \in D$, we have $x \subset \mathrm{cl}_f\, \gamma = \gamma$. Hence $\sup x = \gamma$ by $\sup_{i<\omega} \alpha_i = \sup_{i<\omega} \gamma_{n(i)} = \gamma$.

Fix $k < \omega$. Since $\{\alpha_i : i \le k\} \in T^* \subset T_\delta$, $\mathrm{cl}_f(\{\alpha_i : i \le k\} \cup \delta) \cap \omega_1 = \delta$. So $x \cap \omega_1 = \delta$. Hence $\mathrm{cl}_f(\{\alpha_i : i \le k\} \cup \delta)$ is an initial segment of x by the standard argument using π_γ's. Thus by $\gamma_{n(k)} < \alpha_k$ we have

$$x \cap \gamma_{n(k)} = \mathrm{cl}_f(\{\alpha_i : i \le k\} \cup \delta) \cap \gamma_{n(k)}.$$

Claim. $A \cap x = g(x)$.

Proof. First we claim that $r(x) = \{n(k) : k < \omega\}$. Since $\gamma_{n(k)} < \alpha_k < \gamma_{n(k)+1}$, we have $n(k) \in r(x)$ for every $k < \omega$. To see the converse, fix $k < l < \omega$. Then by the choice of h we have

$$\begin{aligned}\mathrm{cl}_f(\{\alpha_i : i \le l\} \cup \delta) \cap \gamma_{n(k+1)} &\subset \mathrm{cl}_f(\{\alpha_i : i \le l\} \cup \delta) \cap \alpha_{k+1} \\ &\subset h(\{\alpha_i : i \le k\}).\end{aligned}$$

Since $\{\alpha_i : i \le k\} \in T^* \cap [\gamma_{n(k)+1}]^{<\omega}$ and $\gamma_{n(k)+1} \in D$, we have $h(\{\alpha_i : i \le k\}) < \gamma_{n(k)+1}$. Hence $\mathrm{cl}_f(\{\alpha_i : i \le l\} \cup \delta) \cap \gamma_{n(k+1)} \subset \gamma_{n(k)+1}$. Thus $x \cap \gamma_{n(k+1)} \subset \gamma_{n(k)+1}$, as desired.

Therefore $r(x) = \{0\} \cup \bigcup_{k<\omega} J_k$. Since $I_{n(k)}$'s are mutually disjoint, $I_{n(k)} \cap r(x) = J_k$ for every $k < \omega$. Hence for every $k < \omega$

$$\begin{aligned}A \cap x \cap \gamma_{n(k)} &= A \cap \mathrm{cl}_f(\{\alpha_i : i \le k\} \cup \delta) \cap \gamma_{n(k)} \\ &= \psi(\mathrm{cl}_f(\{\alpha_i : i \le k\} \cup \delta) \cap \gamma_{n(k)})(J_k) \\ &= \psi(x \cap \gamma_{n(k)})(I_{n(k)} \cap r(x)).\end{aligned}$$

Thus

$$\begin{aligned} A\cap x &= \bigcup_{k<\omega} A\cap x\cap \gamma_{n(k)} \\ &= \bigcup_{k<\omega} \psi(x\cap\gamma_{n(k)})(I_{n(k)}\cap r(x)) \\ &= \bigcup_{n\in r(x)} \psi(x\cap\gamma_n)(I_n\cap r(x)) \\ &= g(x). \end{aligned}$$

□

This completes the proof. □

Nonetheless we can provide our own proof of Theorem 15, which invokes Theorem 14. Subsequently Koenig and Todorčević [18] figured out their own proof of Theorem 15. which works uniformly for every $\lambda > \omega_1$.

Proof of Theorem 15. By Theorem 1 with $\kappa = \omega_1$ we are done in the case $\lambda > 2^\omega$. So assume $\lambda \leq 2^\omega$. Then $\lambda^\omega = 2^\omega$. We construct a diamond on the set $\{x \in \mathcal{P}_{\omega_1}\lambda : \operatorname{cf}\sup(x\cap\omega_2) = \omega\}$.

Fix a club guessing sequence $\langle c_\gamma : \gamma \in S^\omega_{\omega_2}\rangle$. Let $\{I_n : n<\omega\} \subset [\omega]^\omega$ be a disjoint family with $n < \min I_n$. Since $2^\omega = \lambda^\omega$, we have a map $\varphi : \mathcal{P}(\omega) \to \mathcal{P}_{\omega_1}\lambda$ such that $\varphi``[I_n]^\omega = \mathcal{P}_{\omega_1}\lambda$ for every $n<\omega$.

Fix $x \in \mathcal{P}_{\omega_1}\lambda$ such that $\gamma = \sup(x\cap\omega_2)$ has cofinality ω. List c_γ in increasing order as $\{\gamma_n : n<\omega\}$. Set

$$r(x) = \{n<\omega : x\cap(\gamma_{n+1}-\gamma_n) \neq \emptyset\} \in [\omega]^\omega.$$

Let $g(x) = \bigcup_{n\in r(x)} \varphi(I_n \cap r(x))$. We claim that $g : \mathcal{P}_{\omega_1}\lambda \to \mathcal{P}_{\omega_1}\lambda$ works.

Let $A \subset \lambda$ and $f : [\lambda]^{<\omega} \to \mathcal{P}_{\omega_1}\lambda$. It suffices to give a countable $x \in C(f)$ such that $\operatorname{cf}\sup(x\cap\omega_2) = \omega$ and $A\cap x = g(x)$.

By Lemma 3 we have a stationary subtree T^* of $[S^{\omega_1}_{\omega_2}]^{<\omega}$ and a map $h : T^* \to \omega_2$ such that if $a < b \in T^*$, $(\operatorname{cl}_f b)\cap\min(b-a) \subset h(a)$. Then

$$D = \{\gamma<\omega_2 : (\operatorname{cl}_f\gamma)\cap\omega_2 = \gamma \wedge \forall a \in T^*\cap[\gamma]^{<\omega}(h(a) < \gamma \in \lim \operatorname{suc}_{T^*}(a))\}$$

is club. Take $\gamma \in S^\omega_{\omega_2}\cap D$ with $c_\gamma \subset D-\omega_1$. List c_γ in increasing order as $\{\gamma_n : n<\omega\}$.

By induction on $k<\omega$ we choose $n(k)<\omega$, $\gamma_{n(k)} < \alpha_k < \gamma_{n(k)+1}$ and $J_k \in [I_{n(k)}]^\omega$ so that $n(k) < n(k+1)$, $\{\alpha_i : i<k\} \in T^*$ and $\varphi(J_k) = A\cap\operatorname{cl}_f\{\alpha_i : i\leq k\}$ as follows:

First set $n(0)=0$. Assume next we have $\{\alpha_i : i<k\}$ and $n(k)$ as above. Since $\{\alpha_i : i<k\} \in T^*\cap[\gamma_{n(k)+1}]^{<\omega}$ and $\gamma_{n(k)+1} \in D$, we have $\gamma_{n(k)} < \alpha_k \in \gamma_{n(k)+1}\cap\operatorname{suc}_{T^*}\{\alpha_i : i<k\}$. Then $\{\alpha_i : i\leq k\} \in T^*$.

Since $\varphi``[I_{n(k)}]^\omega = \mathcal{P}_{\omega_1}\lambda$, we have $J_k \in [I_{n(k)}]^\omega$ as required above. Set $n(k+1) = \min(\bigcup_{i\leq k} J_i - (n(k)+1)) > n(k)$.

Note that $n(k) \leq n(l) < \min I_{n(l)} \leq \min J_l$ if $k \leq l < \omega$. Hence $n(k+1) = \min(\bigcup_{i<\omega} J_i - (n(k)+1))$. Thus

$$\{n(k) : k < \omega\} = \{0\} \cup \bigcup_{k<\omega} J_k.$$

Set $x = \bigcup_{k<\omega} \mathrm{cl}_f\{\alpha_i : i \leq k\} \in \mathcal{P}_{\omega_1}\lambda \cap C(f)$. Since $\{\alpha_i : i < \omega\} \subset \gamma \in D$, we have $x \cap \omega_2 \subset \gamma$. Hence $\sup(x \cap \omega_2) = \gamma$ by $\sup_{i<\omega} \alpha_i = \sup_{i<\omega} \gamma_{n(i)} = \gamma$.

Claim. $A \cap x = g(x)$.

Proof. First we claim that $r(x) = \{n(k) : k < \omega\}$. Since $\gamma_{n(k)} < \alpha_k < \gamma_{n(k)+1}$, we have $n(k) \in r(x)$ for every $k < \omega$. To see the converse, fix $k < l < \omega$. Then by the choice of h

$$\begin{aligned}(\mathrm{cl}_f\{\alpha_i : i \leq l\}) \cap \gamma_{n(k+1)} &\subset (\mathrm{cl}_f\{\alpha_i : i \leq l\}) \cap \alpha_{k+1} \\ &\subset h(\{\alpha_i : i \leq k\}).\end{aligned}$$

Since $\{\alpha_i : i \leq k\} \in T^* \cap [\gamma_{n(k)+1}]^{<\omega}$ and $\gamma_{n(k)+1} \in D$, we have $h(\{\alpha_i : i \leq k\}) < \gamma_{n(k)+1}$. Hence $(\mathrm{cl}_f\{\alpha_i : i \leq l\}) \cap \gamma_{n(k+1)} \subset \gamma_{n(k)+1}$. Thus $x \cap \gamma_{n(k+1)} \subset \gamma_{n(k)+1}$, as desired.

Therefore $r(x) = \{0\} \cup \bigcup_{k<\omega} J_k$. Since $I_{n(k)}$'s are mutually disjoint, we have $I_{n(k)} \cap r(x) = J_k$ for every $k < \omega$. Hence for every $k < \omega$ $A \cap \mathrm{cl}_f\{\alpha_i : i \leq k\} = \varphi(J_k) = \varphi(I_{n(k)} \cap r(x))$. Thus

$$\begin{aligned}A \cap x &= \bigcup_{k<\omega} A \cap \mathrm{cl}_f\{\alpha_i : i \leq k\} \\ &= \bigcup_{k<\omega} \varphi(I_{n(k)} \cap r(x)) \\ &= \bigcup_{n\in r(x)} \varphi(I_n \cap r(x)) \\ &= g(x).\end{aligned}$$

□

This completes the proof. □

Our proof of Theorem 15 yields the following generalization:

Theorem 16 (Shioya [26]). *If $2^\omega = 2^{<\kappa}$, $\mathcal{P}_\kappa\lambda$ carries a diamond for every $\lambda > \kappa$.*

The following result shows that some assumption is necessary for Theorem 16.

Theorem 17 (Foreman–Magidor [7]). *It is consistent for $\omega_1 < \kappa < \lambda$ that the club filter on $\mathcal{P}_\kappa\lambda$ is 2^{ω_1}-saturated.*

Theorem 5 follows from Proposition 5 and the following

Theorem 18 (Shioya [26]). *Assume $\omega_1 < \kappa = \kappa^\omega \leq 2^{\omega_1} = 2^{<\kappa}$ and κ-SR in $\mathcal{P}_{\omega_1}2^{\kappa^+}$. Then $\mathcal{P}_\kappa\lambda$ carries a diamond for every $\lambda > \kappa$.*

That $(\kappa^+)^{<\kappa} = (\kappa^+)^{\omega_1}$ holds in the model of Theorem 18 is inevitable by the following

Theorem 19 (Baumgartner [1]). *For every $n < \omega$ $\mathcal{P}_\kappa\kappa^{+n}$ has a club subset of size at most $(\kappa^+)^{\omega_n}$.*

Theorem 18 shows that we have a good control in the case $n = 1$ of Theorem 19. For $n = 2$ even the following question remains open:

Question 5. *Is it consistent that every club subset of $\mathcal{P}_{\omega_3}\omega_5$ has size $\omega_5^{\omega_2} > \omega_5^{\omega_1}$?*

See [26] for a proof of Theorem 18. Instead we present a proof of the following result, which combines those of Theorems 15 and 18:

Proposition 11. *Assume $\omega_1 < \kappa = \kappa^\omega \leq 2^{\omega_1} = 2^{<\kappa}$ and κ-SR in $\mathcal{P}_{\omega_1}2^{\kappa^+}$. Then the set $\{x \in \mathcal{P}_\kappa\kappa^+ : \mathrm{cf}(x \cap \kappa) = \omega \wedge \mathrm{cf}\sup x = \omega_1\}$ carries a diamond.*

In the case of $2^{<\kappa} = \kappa$ this gives a new example because Theorem 14 produces one on the set $\{x \in \mathcal{P}_\kappa\kappa^+ : \mathrm{cf}\sup x = \omega\}$.

Proof of Proposition 11. By Lemma 1 we have a club guessing sequence $\langle c_\gamma : \gamma \in S^{\omega_1}_{\kappa^+}\rangle$ such that $c_\gamma \subset S^\kappa_{\kappa^+}$ if $\gamma \in S^{\omega_1}_{\kappa^+} \cap \lim S^\kappa_{\kappa^+}$. Fix a disjoint family $\{I_\eta : \eta < \omega_1\} \subset [\omega_1]^{\omega_1}$ with $\eta < \min I_\eta$. Since $2^{\omega_1} = 2^{<\kappa}$, we have a map $\psi(z) : \mathcal{P}(\omega_1) \to \mathcal{P}(z)$ such that $\psi(z)``[I_\eta]^{\omega_1} = \mathcal{P}(z)$ for every $\eta < \omega_1$.

Fix $x \in \mathcal{P}_\kappa\kappa^+$ such that $\gamma = \sup x$ has cofinality ω_1. List c_γ in increasing order as $\{\gamma_\eta : \eta < \omega_1\}$. Set

$$r(x) = \{\eta < \omega_1 : x \cap (\gamma_{\eta+1} - \gamma_\eta) \neq \emptyset\}.$$

(Note that $r(x) = \emptyset$ for some x.) Let $g(x) = \bigcup_{\eta \in r(x)} \psi(x \cap \gamma_\eta)(I_\eta \cap r(x))$. We claim that $g : \mathcal{P}_\kappa\kappa^+ \to \mathcal{P}_\kappa\kappa^+$ works.

Fix $A \subset \kappa^+$ and $f : [\kappa^+]^{<\omega} \to \mathcal{P}_{\omega_1}\kappa^+$. It suffices to give $x \in C(f)$ of size $< \kappa$ such that $\mathrm{cf}(x \cap \kappa) = \omega$, $\mathrm{cf}\sup x = \omega_1$ and $A \cap x = g(x)$.

Set $\theta = 2^{\kappa^+}$. List the functions : $[\theta]^{<\omega} \to \mathcal{P}_{\omega_1}\theta$ as $\{e_\beta : \beta < 2^\theta\}$. For $\kappa \leq \gamma < \kappa^+$ fix a bijection $\pi_\gamma : \kappa \to \gamma$. Then

$$D = \{z \in \mathcal{P}_\kappa 2^\theta : z \cap \kappa^+ \in C(f) \wedge \forall\beta \in z(z \cap \theta \in C(e_\beta)) \wedge \\ \forall\gamma \in z \cap (\kappa^+ - \kappa)(\pi_\gamma``(z \cap \kappa) = z \cap \gamma)\}$$

is σ-club. Lemma 3 provides us $d : [2^\theta]^{<\omega} \to \mathcal{P}_{\omega_1}2^\theta$ such that if $z \in \mathcal{P}_\kappa 2^\theta \cap C(d)$ and $u \in \mathcal{P}_\kappa \kappa^+$, $z \in D$ and $\mathrm{cl}_d(z \cup u) \cap \theta = \mathrm{cl}_d((z \cap \theta) \cup u) \cap \theta$.

Set

$$C = \{x \in \mathcal{P}_{\omega_1}\theta : \{\alpha \in S^\kappa_{\kappa^+} : \mathrm{cl}_d(x \cup \{\alpha\} \cup \sup(x \cap \kappa)) \cap \kappa = \sup(x \cap \kappa)\} \text{ is stationary in } \kappa^+\}.$$

Claim. *C has a club subset.*

Proof. Let S be stationary in $\mathcal{P}_{\omega_1}\theta$. We show $S \cap C \neq \emptyset$.

By κ-SR we have $\kappa \subset X \subset \theta$ of size κ such that $S \cap \mathcal{P}_{\omega_1}X$ is stationary in $\mathcal{P}_{\omega_1}X$. Fix a bijection $\pi : \kappa \to X$. Since $\{x \in \mathcal{P}_{\omega_1}X : \pi``(x \cap \kappa) = x\}$ is club, $\{x \in S \cap \mathcal{P}_{\omega_1}X : \pi``(x \cap \kappa) = x\}$ is stationary in $\mathcal{P}_{\omega_1}X$. Hence $S' = \{\sup(x \cap \kappa) : \pi``(x \cap \kappa) = x \in S \cap \mathcal{P}_{\omega_1}X\}$ is stationary in κ by $\kappa \subset X$.

Since $\{z \in \mathcal{P}_\kappa 2^\theta \cap C(d) : \pi``(z \cap \kappa) \subset z\}$ is club in $\mathcal{P}_\kappa 2^\theta$,

$$S'' = \{z \in \mathcal{P}_\kappa 2^\theta \cap C(d) : \pi``(z \cap \kappa) \subset z \wedge z \cap \kappa \in S'\}$$

is stationary in $\mathcal{P}_\kappa 2^\theta$. Hence $\bigcup S'' = 2^\theta$. Since $\{z \cap \kappa : z \in S''\} \subset S' \subset \kappa$, we have $\delta \in S'$ such that $S^* = S^\kappa_{\kappa^+} \cap \bigcup\{z \in S'' : z \cap \kappa = \delta\}$ is stationary in κ^+. Since $\delta \in S'$, we have $x \in S$ such that $\pi``(x \cap \kappa) = x$ and $\sup(x \cap \kappa) = \delta$. We claim that $x \in C$. It suffices to show $S^* \subset \{\alpha \in S^\kappa_{\kappa^+} : \mathrm{cl}_d(x \cup \{\alpha\} \cup \delta) \cap \kappa = \delta\}$.

Fix $\alpha \in S^*$. We have $z \in S''$ with $\alpha \in z$ and $z \cap \kappa = \delta$. Then $x = \pi``(x \cap \kappa) \subset \pi``\sup(x \cap \kappa) = \pi``\delta = \pi``(z \cap \kappa) \subset z$. Since $x \cup \{\alpha\} \cup \delta \subset z \in C(d)$, $\delta \subset \mathrm{cl}_d(x \cup \{\alpha\} \cup \delta) \cap \kappa \subset z \cap \kappa = \delta$. Hence $\mathrm{cl}_d(x \cup \{\alpha\} \cup \delta) \cap \kappa = \delta$, as desired. □

Now we have $\beta < 2^\theta$ with $\mathcal{P}_{\omega_1}\theta \cap C(e_\beta) \subset C$. Take a countable $y \in C(d)$ with $\beta \in y$. Then $\delta = \sup(y \cap \kappa)$ has cofinality ω. Since $\beta \in y \in \mathcal{P}_{\omega_1}2^\theta \cap C(d) \subset D$, $y \cap \theta \in \mathcal{P}_{\omega_1}\theta \cap C(e_\beta) \subset C$. Hence $\mathrm{cl}_d((y \cap \theta) \cup \{\alpha\} \cup \delta) \cap \kappa = \delta$ for some $\alpha < \kappa^+$. So by the choice of d $\delta \subset \mathrm{cl}_d(y \cup \delta) \cap \kappa \subset \mathrm{cl}_d(y \cup \{\alpha\} \cup \delta) \cap \kappa = \mathrm{cl}_d((y \cap \theta) \cup \{\alpha\} \cup \delta) \cap \kappa = \delta$. Thus $\mathrm{cl}_d(y \cup \delta) \cap \kappa = \delta$.

Define a tree order $\leq$ on $[\kappa^+]^{<\omega_1}$ by end-extension. By a subtree of $[\kappa^+]^{<\omega_1}$ we mean a subset T closed under initial segments such that if $b \in [\kappa^+]^{<\omega_1}$ has limit order type and $a \in T$ for every $a < b$, $b \in T$. Define a subtree of $[\kappa^+]^{<\omega_1}$ to be stationary as in §2.

Set $T = \{a \in [S^\kappa_{\kappa^+}]^{<\omega_1} : \mathrm{cl}_d(y \cup a \cup \delta) \cap \kappa = \delta\}$. It is easy to check that T is a nonempty subtree of $[\kappa^+]^{<\omega_1}$.

Claim. *There are a stationary subtree T^* of T and $h : T^* \to \kappa^+$ such that if $a < b \in T^*$, $\mathrm{cl}_d(y \cup b \cup \delta) \cap \min(b - a) \subset h(a)$.*

Proof. First we claim that T is a stationary subtree. Fix $a \in T$. It suffices to show that $\{\alpha \in S^{\kappa}_{\kappa^+} : \mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta) \cap \kappa = \delta\}$ is stationary in κ^+.

Set $z = \mathrm{cl}_d(y \cup a)$. Then $y \subset z \subset \mathrm{cl}_d(y \cup a \cup \delta)$. Hence by $a \in T$ we have $\delta = \sup(y \cap \kappa) \subset \sup(z \cap \kappa) \subset \sup(\mathrm{cl}_d(y \cup a \cup \delta) \cap \kappa) = \delta$. Thus $\sup(z \cap \kappa) = \delta$. Since $\beta \in z \in \mathcal{P}_{\omega_1} 2^{\theta} \cap C(d) \subset D$, $z \cap \theta \in \mathcal{P}_{\omega_1}\theta \cap C(e_\beta) \subset C$. Hence $\{\alpha \in S^{\kappa}_{\kappa^+} : \mathrm{cl}_d((z \cap \theta) \cup \{\alpha\} \cup \delta) \cap \kappa = \delta\}$ is stationary in κ^+. Fix α from this set. Then by the choice of d we have $\delta \subset \mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta) \cap \kappa \subset \mathrm{cl}_d(z \cup \{\alpha\} \cup \delta) \cap \kappa = \mathrm{cl}_d((z \cap \theta) \cup \{\alpha\} \cup \delta) \cap \kappa = \delta$. Hence $\mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta) \cap \kappa = \delta$, as desired.

Fix $a \in T$. The map $\alpha \mapsto \sup(\mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta) \cap \alpha)$ is regressive on $\mathrm{suc}_T(a)$ by $|\,\mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta)| < \kappa = \mathrm{cf}\,\alpha$. Hence we have a stationary $S_a \subset \mathrm{suc}_T(a)$ and $h(a) < \kappa^+$ such that $\mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta) \cap \alpha \subset h(a)$ for every $\alpha \in S_a$.

Define inductively a stationary subtree T^* of T by $\mathrm{suc}_{T^*}(a) = S_a$ for $a \in T^*$. We show that T^* and $h|T^*$ work.

Let $a < b \in T^*$ and $\alpha = \min(b - a)$. Since $a \cup \{\alpha\} \in T^*$, we have $\mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta) \cap \kappa = \mathrm{cl}_d(y \cup b \cup \delta) \cap \kappa = \delta$. Hence $\mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta) \cap \kappa^+$ is an initial segment of $\mathrm{cl}_d(y \cup b \cup \delta) \cap \kappa^+$ by the standard argument using π_γ's. Thus $\mathrm{cl}_d(y \cup b \cup \delta) \cap \alpha = \mathrm{cl}_d(y \cup a \cup \{\alpha\} \cup \delta) \cap \alpha \subset h(a)$, as desired. □

Define

$$E = \{\gamma < \kappa^+ : \mathrm{cl}_d(y \cup \gamma) \cap \kappa^+ = \gamma \wedge \\ \forall a \in T^* \cap [\gamma]^{<\omega_1} (h(a) < \gamma \in \lim \mathrm{suc}_{T^*}(a))\}.$$

Since $\kappa^\omega = \kappa$, E is unbounded. Hence $\lim E$ is club in κ^+. Take $\gamma \in S^{\omega_1}_{\kappa^+} \cap \lim S^{\kappa}_{\kappa^+} \cap \lim E$ with $c_\gamma \subset \lim E$. Then $c_\gamma \subset S^{\kappa}_{\kappa^+}$. Note that $\{\gamma < \kappa^+ : \mathrm{cf}\,\gamma > \omega\} \cap \lim E \subset E$. Hence $c_\gamma \cup \{\gamma\} \subset E$ by the definition of E. List c_γ in increasing order as $\{\gamma_\eta : \eta < \omega_1\}$.

By induction on $\zeta < \omega_1$ we choose $\eta(\zeta) < \omega_1$, $\gamma_{\eta(\zeta)} < \alpha_\zeta < \gamma_{\eta(\zeta)+1}$ and $J_\zeta \in [I_{\eta(\zeta)}]^{\omega_1}$ so that $\{\eta(\iota) : \iota \leq \zeta\}$ is increasing, $\{\alpha_\iota : \iota \leq \zeta\} \in T^*$ and $\psi(\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)})(J_\zeta) = A \cap \mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)}$ as follows:

Assume we have $\eta(\iota)$, α_ι and J_ι for $\iota < \zeta$ as above. Set

$$\eta(\zeta) = \min\Big(\bigcup_{\iota<\zeta} J_\iota - \sup_{\iota<\zeta}(\eta(\iota) + 1)\Big)$$

if $\zeta > 0$. We stipulate $\eta(0) = 0$. Since $\{\alpha_\iota : \iota < \zeta\} \in T^* \cap [\gamma_{\eta(\zeta)+1}]^{<\omega_1}$ and $\gamma_{\eta(\zeta)+1} \in E$, we have $\gamma_{\eta(\zeta)} < \alpha_\zeta \in \gamma_{\eta(\zeta)+1} \cap \mathrm{suc}_{T^*}\{\alpha_\iota : \iota < \zeta\}$. Then $\{\alpha_\iota : \iota \leq \zeta\} \in T^*$. Since $\psi(\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)})\text{“}[I_{\eta(\zeta)}]^{\omega_1} =$

$\mathcal{P}(\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)})$, we have $J_\zeta \in [I_{\eta(\zeta)}]^{\omega_1}$ as required above.

Note that $\eta(\zeta) \leq \eta(\xi) < \min I_{\eta(\xi)} \leq \min J_\xi$ if $\zeta \leq \xi < \omega_1$. Hence $\eta(\zeta) = \min(\bigcup_{\iota<\omega_1} J_\iota - \sup_{\iota<\zeta}(\eta(\iota)+1))$ for every $\zeta > 0$. Thus

$$\{\eta(\zeta) : \zeta < \omega_1\} = \{0\} \cup \bigcup_{\zeta<\omega_1} J_\zeta.$$

Now set $x = \bigcup_{\zeta<\omega_1} \mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \kappa^+$. Then $x \in \mathcal{P}_\kappa\lambda \cap C(f)$ by $\bigcup_{\zeta<\omega_1} \mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \in \mathcal{P}_\kappa 2^\theta \cap C(d) \subset D$. Also we have $x \cap \kappa = \bigcup_{\zeta<\omega_1} \mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \kappa = \delta$ by $\{\alpha_\iota : \iota \leq \zeta\} \in T^* \subset T$. Since $\{\alpha_\iota : \iota < \omega_1\} \cup \delta \subset \gamma \in E$, we have $x \subset \mathrm{cl}_d(y \cup \gamma) \cap \kappa^+ = \gamma$. Hence $\sup x = \gamma$ by $\sup_{\iota<\omega_1} \alpha_\iota = \sup_{\iota<\omega_1} \gamma_{\eta(\iota)} = \gamma$.

Fix $\zeta < \omega_1$. Since $\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \kappa = x \cap \kappa = \delta$, $\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \kappa^+$ is an initial segment of x by the standard argument using π_γ's. Hence by $\gamma_{\eta(\zeta)} < \alpha_\zeta$ we have

$$x \cap \gamma_{\eta(\zeta)} = \mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)}.$$

Claim. $A \cap x = g(x)$.

Proof. First we claim that $r(x) = \{\eta(\zeta) : \zeta < \omega_1\}$. Since $\gamma_{\eta(\zeta)} < \alpha_\zeta < \gamma_{\eta(\zeta)+1}$, we have $\eta(\zeta) \in r(x)$ for every $\zeta < \omega_1$. To see the converse, fix $\zeta < \xi < \omega_1$. Then by the choice of h

$$\begin{aligned}\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \xi\} \cup \delta) \cap \gamma_{\eta(\zeta+1)} &\subset \mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \xi\} \cup \delta) \cap \alpha_{\zeta+1} \\ &\subset h(\{\alpha_\iota : \iota \leq \zeta\}).\end{aligned}$$

Since $\{\alpha_\iota : \iota \leq \zeta\} \in T^* \cap [\gamma_{\eta(\zeta)+1}]^{<\omega_1}$ and $\gamma_{\eta(\zeta)+1} \in E$, we have $h(\{\alpha_\iota : \iota \leq \zeta\}) < \gamma_{\eta(\zeta)+1}$. Hence $\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \xi\} \cup \delta) \cap \gamma_{\eta(\zeta+1)} \subset \gamma_{\eta(\zeta)+1}$. Thus $x \cap \gamma_{\eta(\zeta+1)} \subset \gamma_{\eta(\zeta)+1}$, as desired.

Therefore $r(x) = \{0\} \cup \bigcup_{\zeta<\omega_1} J_\zeta$. Since $I_{\eta(\zeta)}$'s are mutually disjoint, $I_{\eta(\zeta)} \cap r(x) = J_\zeta$ for every $\zeta < \omega_1$. Hence for every $\zeta < \omega_1$

$$\begin{aligned}&A \cap \mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)} \\ &= \psi(\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)})(J_\zeta) \\ &= \psi(\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)})(I_{\eta(\zeta)} \cap r(x)).\end{aligned}$$

Thus

$$\begin{aligned}
A \cap x &= \bigcup_{\zeta<\omega_1} A \cap x \cap \gamma_{\eta(\zeta)} \\
&= \bigcup_{\zeta<\omega_1} A \cap \mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)} \\
&= \bigcup_{\zeta<\omega_1} \psi(\mathrm{cl}_d(y \cup \{\alpha_\iota : \iota \leq \zeta\} \cup \delta) \cap \gamma_{\eta(\zeta)})(I_{\eta(\zeta)} \cap r(x)) \\
&= \bigcup_{\zeta<\omega_1} \psi(x \cap \gamma_{\eta(\zeta)})(I_{\eta(\zeta)} \cap r(x)) \\
&= \bigcup_{\eta\in r(x)} \psi(x \cap \gamma_\eta)(I_\eta \cap r(x)) \\
&= g(x).
\end{aligned}$$

□

This completes the proof. □

References

[1] Baumgartner, J., *On the size of closed unbounded sets*, Ann. Pure Appl. Logic 54 (1991) 195–227.

[2] Baumgartner, J. and Taylor, A., *Saturation properties of ideals in generic extensions. II*, Trans. Amer. Math. Soc. 271 (1982) 587–609.

[3] Bekkali, M., Topics in Set Theory, Lecture Notes in Math. 1476, Springer, Berlin, 1991.

[4] Burke, D. and Matsubara, Y., *The extent of strength in the club filters*, Israel J. Math. 114 (1999) 253–263.

[5] Donder, H.-D. and Levinski, J.-P., *Some principles related to Chang's conjecture*, Ann. Pure Appl. Logic 45 (1989) 39–101.

[6] Donder, H.-D. and Matet, P., *Two cardinal versions of diamond*, Israel J. Math. 83 (1993) 1–43.

[7] Foreman, M. and Magidor, M., *Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $P_\kappa(\lambda)$*, Acta Math. 186 (2001) 271–300.

[8] Foreman, M., Magidor, M. and Shelah, S., *Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part I*, Ann. Math. 127 (1988) 1–47.

[9] Foreman, M. and Todorčević, S., *A new Löwenheim–Skolem theorem*, Trans. Amer. Math. Soc. 357 (2005) 1693–1715.

[10] Gitik, M., *Some results on the nonstationary ideal*, Israel J. Math. 92 (1995) 61–112.

[11] Gitik, M. and Shelah, S., *Less saturated ideals*, Proc. Amer. Math. Soc. 125 (1997) 1523–1530.

[12] ———, *Cardinal preserving ideals*, J. Symbolic Logic 64 (1999) 1527–1551.

[13] Goldring, N., *Woodin cardinals and presaturated ideals*, Ann. Pure Appl. Logic 55 (1992) 285–303.

[14] ———, *The entire NS ideal on $\mathcal{P}_\gamma\mu$ can be precipitous*, J. Symbolic Logic 62 (1997) 1161–1172.

[15] Jech, T., *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic 5 (1973) 165–198.

[16] Jech, T. and Prikry, K., *Ideals over uncountable sets: application of almost disjoint functions and generic ultrapowers*, Mem. Amer. Math. Soc. 214 (1979).

[17] Kanamori, A., The Higher Infinite, Springer Monogr. in Math., Springer, Berlin, 2003.

[18] Koenig, B. and Todorčević, S., *Diamond indexed by countable sets*, typed notes.

[19] Shelah, S., Proper Forcing, Lecture Notes in Math. 940, Springer, Berlin, 1982.

[20] ———, Around Classification Theory of Models, Lecture Notes in Math. 1182, Springer, Berlin, 1986.

[21] ———, Cardinal Arithmetic, Oxford Logic Guides 29, Oxford Univ. Press, New York, 1994.

[22] ———, Nonstructure Theory, to be published.

[23] Shelah, S. and Shioya, M., *Nonreflecting stationary sets in $\mathcal{P}_\kappa\lambda$*, Adv. Math. 199 (2006) 185–191.

[24] Shioya, M., *Splitting $\mathcal{P}_\kappa\lambda$ into maximally many stationary sets*, Israel J. Math. 114 (1999) 347–357.

[25] ———, *Stationary reflection and the club filter*, preprint.

[26] ———, *Diamonds on $\mathcal{P}_\kappa\lambda$*, Proceedings of Computational Prospects of Infinity, to appear.

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